STABILITY IN FIRST APPROXIMATION OF SYSTEMS WITH LAG

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Certain results are established in the theory of stability on the basis of first approximations for systems with lag [delayed systems]. Sufficient conditions are obtained for first approximation stability of such systems.

1. Let us consider the following system of equations of perturbed motion:

$$\frac{dx_i}{dt} = F_i(t; x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau)) \qquad (i = 1, \ldots, n) \qquad (1.1)$$

where $F(t; x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau))$ are holomorphic functions of the variables $x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau)$, satisfying the conditions $F_i(t; 0, \ldots, 0, \ldots, 0) = 0$ $(i = 1, \ldots, n)$.

Expanding the right-hand terms of the equations (1.1) in powers of the variables $x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau)$, we obtain

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} (p_{ij}(t)\, x_j(t) + q_{ij}(t)\, x_j(t-\tau)) + X_i \qquad (i = 1, \ldots, n) \qquad (1.2)$$

where $p_{ij}(t)$ and $q_{ij}(t)$ stand for $\partial F_i/\partial x_j(t)$ and $\partial F_i/\partial x_j(t-\tau)$, respectively, when $x_j(t) = 0$, $x_j(t-\tau) = 0$; the $X_i (i = 1, \ldots, n)$ are functions whose power expansions begin with terms of degree not lower than the second.

Along with the system of equations of the first approximation

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} (p_{ij}(t)\, x_j(t) + q_{ij}(t)\, x_j(t-\tau)) \qquad (i = 1, \ldots, n) \qquad (1.3)$$

we shall consider the system

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$$\frac{dx_i}{dt} = \sum_{j=1}^{n} (p_{ij}(t) + q_{ij}(t))\, x_j \qquad (i = 1, \ldots, n) \tag{1.4}$$

which is obtained from (1.3) by setting $\tau = 0$.

Let us suppose that the null solution (i.e. the identically vanishing or trivial solution) of the system (1.4) is asymptotically stable, and that for it there is given a positive-definite quadratic form, with bounded coefficients,

$$V = \sum_{i,\, j=1}^{n} \alpha_{ij}(t)\, x_i x_j$$

which satisfies the hypotheses of Liapunov's theorem [1] on asymptotic stability. Under these conditions and by equation (1.4), the derivative of V is given by

$$\frac{dV}{dt} = \sum_{i,\, j=1}^{n} a_{ij}^{(0)}(t)\, x_i x_j \tag{1.5}$$

where

$$a_{ij}^{(0)}(t) = \sum_{s=1}^{n} [\alpha_{is}(t)(p_{sj}(t) + q_{sj}(t)) + \alpha_{js}(t)(p_{si}(t) + q_{si}(t))] + \frac{d\alpha_{ij}}{dt}$$

$$(i, j = 1, \ldots, n)$$

Furthermore, the expression in (1.5) is a negative-definite quadratic form.

We shall next explain under what conditions the quadratic form V will be a function satisfying Liapunov's theorem on asymptotic stability for a system of differential equations of first approximations of the type (1.3). The derivative of the quadratic form V is, because of (1.3).

$$\frac{dV}{dt} = \sum_{i;\, j=1}^{n} b_{ij}(t)\, x_i(t)\, x_j(t) + \sum_{i,\, j=1}^{n} c_{ij}(t)\, x_i(t)\, x_j(t-\tau) \tag{1.6}$$

where

$$b_{ij}(t) = \sum_{s=1}^{n} (\alpha_{is}(t)\, p_{sj}(t) + \alpha_{js}(t)\, p_{si}(t)) + \frac{d\alpha_{ij}}{dt}$$

$$c_{ij}(t) = 2\sum_{s=1}^{n} \alpha_{is}(t)\, q_{sj}(t)$$

For the purpose of simplifying later computations, we make a change of variables and reduce the quadratic form V to the sum of squares, $V = y_1^2 + \ldots + y_n^2$.

It is well known that this can always be accomplished for definite forms by a non-singular linear transformation of the type

$$x_i = k_{i1}(t)\, y_1 + \ldots + k_{in}(t)\, y_n$$

Making the indicated change of variables in (1.6), we obtain

$$\frac{d}{dt}\Big(\sum_{i=1}^{n} y_i^2\Big) = \sum_{i;\, j=1}^{n} A_{ij}(t)\, y_i(t)\, y_j(t) + \sum_{i;\, j=1}^{n} B_{ij}(t)\, y_i(t)\, y_j(t-\tau) \quad (1.7)$$

where

$$A_{ij}(t) = \sum_{\beta;\, \gamma=1}^{n} b_{\beta\gamma} k_{\beta i}(t)\, k_{\gamma j}(t), \qquad B_{ij}(t) = \sum_{\beta;\, \gamma=1}^{n} c_{\beta\gamma} k_{\beta i}(t)\, k_{\gamma j}(t-\tau)$$

On the basis of Theorem 5 proved in [2], the null solution of the system (1.3) will be asymptotically stable if there exists a positive-definite function $V(t;\ x_1,\ \ldots,\ x_n)$, which has an arbitrarily small upper bound, and which is such that its derivative is negative-definite for all values $x_1(t-\tau),\ \ldots,\ x_n(t-\tau)$ satisfying the condition

$$V(t-\tau,\ x_1(t;\ -\tau), \ldots, x_n(t-\tau)) \leqslant V(t;\ x_1(t), \ldots, x_n(t)) \quad (1.8)$$

This condition (1.8) takes the following form in terms of the new variables $y_1,\ \ldots,\ y_n$:

$$y_1^{(2)}(t-\tau) + \ldots + y_n^2(t-\tau) \leqslant y_1^2(t) + \ldots + y_n^2(t) \quad (1.9)$$

Next, let us determine the maximum of the function dV/dt for fixed values $y_1(t),\ \ldots,\ y_n(t)$ and for values $y_1(t-\tau),\ \ldots,\ y_n(t-\tau)$ satisfying the condition (1.9). Let us set $y_i(t-\tau) = z_i$ $(i = 1,\ \ldots,\ n)$

Using Lagrange's method [3], we construct the function

$$\Phi = \frac{dV}{dt} - \lambda\, [V(z_1, \ldots, z_n) - V(y_1, \ldots, y_n)]$$

or by (1.7)

$$\Phi = \sum_{i;\, j=1}^{n} A_{ij}(t)\, y_i y_j + \sum_{i;\, j=1}^{n} B_{ij}(t)\, y_i z_j + \lambda \Big[\sum_{i=1}^{n} z_i^2 - \sum_{i=1}^{u} y_i^2\Big] \quad (1.10)$$

The values $z_1,\ \ldots,\ z_n$ for which the function dV/dt takes on a maximum are found from the system of equations

$$\frac{\partial \Phi}{\partial z_j} = \sum_{i=1}^{n} B_{ij}(t)\, y_i - 2\lambda z_j = 0, \qquad \sum_{i=1}^{n} (z_i^2 - y_i^2) = 0 \quad (1.11)$$

By determining the z_j from the first set of equations, and substituting their values in the second set, we find

$$\lambda = \frac{1}{2}\Big[\sum_{j=1}^{n}\Big(\sum_{i=1}^{n} B_{ij}(t)\, y_i\Big)^2 \Big/ \sum_{j=1}^{n} y_j^2\Big]^{1/2} \quad (1.12)$$

Multiplying each of the first n equations (1.11) by z_j and adding the result, we obtain

$$\sum_{i,\,j=1}^{n} B_{ij}(t)\, y_i z_j - 2\lambda \sum_{j=1}^{n} z_j^2 = 0 \tag{1.13}$$

But from the second equation of (1.11) it follows that $z_1^2 + \ldots + z_n^2 = y_1^2 + \ldots + y_n^2$, and hence,

$$\sum_{i,\,j=1}^{n} B_{ij}(t)\, y_i z_j = \left\{\left(\sum_{j=1}^{n} y_j^2\right)\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n} B_{ij}(t)\, y_i\right)^2\right]\right\}^{1/2} \tag{1.14}$$

The quadratic form appearing in the square brackets of equation (1.14) can be transformed to its canonical form by means of an orthogonal transformation. If this is done we obtain

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n} B_{ij}(t)\, y_i\right)^2 = \sum_{i=1}^{n} h_i(t)\, \xi_i^2 \tag{1.15}$$

where $\xi_1, \ldots, \xi_n$ are linear functions of the variables $y_1, \ldots, y_n$ such that $y_1^2 + \ldots + y_n^2 = \xi_1^2 + \ldots + \xi_n^2$. Making the indicated substitutions in (1.14) we obtain

$$\sum_{i,\,j=1}^{n} B_{ij}(t)\, y_i z_j = \left[\left(\sum_{i=1}^{n} \xi_i^2\right)\left(\sum_{i=1}^{n} h_i(t)\, \xi_i^2\right)\right]^{1/2} \tag{1.16}$$

Let $h'(t) = \sup\{h_1(t), \ldots, h_n(t)\}$, $h''(t) = \inf\{h_1(t), \ldots, h_n(t)\}$. Then the following inequality holds [5]

$$h''(t)\left(\sum_{i=1}^{n} \xi_i^2\right) \leqslant \sum_{i=1}^{n} h_i(t)\, \xi_i^2 \leqslant \left(\sum_{i=1}^{n} \xi_i^2\right) h'(t) \tag{1.17}$$

On the basis of (1.16) and (1.7) it can be shown that the bilinear form which appears on the left-hand side of equation (1.14) satisfies the inequality

$$\sqrt{h''(t)} \sum_{i=1}^{n} \xi_i^2 \leqslant \sum_{i,\,j=1}^{n} B_{ij}(t)\, y_i z_j \leqslant \sqrt{h'(t)} \sum_{i=1}^{n} \xi_i^2 \tag{1.18}$$

if $z_1^2 + \ldots + z_n^2 < y_1^2 + \ldots + y_n^2$.

Introducing the variables $\xi_1, \ldots, \xi_n$ into equation (1.7) and taking into account the fact that $y_1^2 + \ldots + y_n^2 = \xi_1^2 + \ldots + \xi_n^2$, we obtain

$$\frac{d}{dt}\left(\sum_{i=1}^{n} \xi_i^2(t)\right) = \sum_{i;\,j=1}^{n} \beta_{ij}(t)\, \xi_i(t)\, \xi_j(t) + \sum_{i,\,j=1}^{n} \gamma_{ij}(t)\, \xi_i(t)\, \xi_j(t-\tau) \tag{1.19}$$

Under the condition that $\xi_1^2\,(t-\tau) + \ldots + \xi_n^2\,(t-\tau) < \xi_1^2\,(t) + \ldots + \xi_n^2\,(t)$ we have

$$\sqrt{h''(t)}\sum_{i=1}^{n}\xi_i^2(t)\leqslant\sum_{i;\,j=1}^{n}\gamma_{ij}(t)\,\xi_i(t)\,\xi_j(t-\tau)\leqslant\sqrt{h'(t)}\sum_{i=1}^{n}\xi_i^2(t)\qquad(1.20)$$

On the basis of (1.20) we can obtain an estimate for the derivative dV/dt, namely,

$$\sum_{i;\,j=1}^{n}(\beta_{ij}(t)+\delta_{ij}\sqrt{h'(t)})\,\xi_i(t)\,\xi_j(t)\geqslant$$

$$\geqslant\frac{dV}{dt}=\frac{d}{dt}\sum_{i=1}^{n}\xi_i^2(t)\geqslant\sum_{i;\,j=1}^{n}(\beta_{ij}(t)+\delta_{ij}\sqrt{h''(t)})\,\xi_i(t)\,\xi_j(t)\qquad(1.21)$$

$$(\delta_{ij}=0\ \text{when}\ i\neq j,\ \delta_{ii}=1),\quad V=\xi_1^2+\ldots+\xi_n^2$$

From this inequality (1.21) we can now draw the following result.

Theorem 1. The trivial solution of the system of differential equations of first approximations is asymptotically stable for all values of the lagging argument τ if the roots $\lambda_1(t)$, ..., $\lambda_n(t)$ of the equation

$$\det\|\beta_{ij}(t)+\delta_{ij}(\sqrt{h'(t)}-\lambda)\|=0$$

satisfy the condition $\lambda_i(t)\leqslant\epsilon<0$ $(i=1,\ldots,n)$ $t>t_0$, where ϵ is a fixed negative number of arbitrarily small absolute value.

2. Considerably less rigid conditions for stability can be obtained on the basis of Theorems 6 and 7 of reference [2].

Because of (1.3) one may write the expression (1.6) for the derivative of V in the following form

$$\frac{dV}{dt}=\sum_{i,\,j=1}^{n}a_{ij}^0(t)\,x_i(t)\,x_j(t)+\sum_{i,\,j=1}^{n}c_{ij}(t)\,x_i(t)\,[x_j(t-\tau)-x_j(t)]\qquad(2.1)$$

where the first sum on the right-hand side is the negative-definite derivative of V in consequence of (1.4).

By a well-known formula of Lagrange for finite increments

$$x_j(t-\tau)-x_j(t)=-\tau\frac{d}{dt}x_j(\sigma_j)\qquad(\sigma_j=t-\theta_j\tau,\ 0<\theta_j<1)\qquad(2.2)$$

From the equations of the first approximation (1.3) we obtain

$$x_j(t-\tau)-x_j(t)=-\tau\sum_{s=1}^{n}[p_{js}(\sigma_j)\,x_s(\sigma_j)+q_{js}(\sigma_j)\,x_s(\sigma_j-\tau)]\quad(j=1,\ldots,n)\qquad(2.3)$$

Making the appropriate substitutions in equation (2.1), we get

$$\frac{dV}{dt}=\sum_{i;\,j=1}^{n}a_{ij}^0(t)\,x_i(t)\,x_j(t)-\tau\sum_{i=1}^{n}x_i(t)\sum_{j=1}^{n}c_{ij}(t)\times$$

$$\times\sum_{s=1}^{n}[p_{js}(\sigma_j)\,x_s(\sigma_j)+q_{js}(\sigma_j)\,x_s(\sigma_j-\tau)]\qquad(2.4)$$

On the basis of theorems proved in [2], we have the following result. The stability of the null solution of a system of differential equations of the first approximation (1.3) follows from the negativeness of the function dV/dt along every integral curve satisfying the condition

$$V(\sigma, x_1(\sigma), \ldots, x_n(\sigma)) \leqslant V(t, x_1(t), \ldots, x_n(t)) \quad \text{where } \sigma \leqslant t \tag{2.5}$$

Making use of the linear transformation

$$x_i = k_{i1}(t) y_1 + \ldots + k_{in}(t) y_n$$

where the $y_1, \ldots, y_n$ are such that $V = y_1^2 + \ldots + y_n^2$, we obtain

$$\frac{d}{dt}\sum_{i=1}^{n} y_i^2(t) = \sum_{i;\, j=1}^{n} a_{ij}^{(1)}(t) y_i(t) y_j(t) - U\tau$$
$$a_{ij}^{(1)}(t) = \sum_{\mu;\, \nu=1}^{n} a_{\mu\nu}^{0}(t) k_{\mu i}(t) k_{\nu j}(t) \tag{2.6}$$

$$U = \sum_{i;\, j,\, \mu,\, \nu,\, s=1}^{n} c_{\mu\nu}(t) k_{\mu i}(t) y_i(t) [p_{\nu s}(\sigma_\nu) k_{sj}(\sigma_\nu) y_j(\sigma_\nu) + q_{\nu s}(\sigma_\nu) k_{sj}(\sigma_\nu - \tau) y_j(\sigma_\nu - \tau)] \tag{2.7}$$

The stability of the system of first approximations is implied by the negativeness of the left-hand side of equation (2.6) under condition (2.5) or under the equivalent conditions

$$\sum_{i=1}^{n} y_i^2(\sigma_j) \leqslant \sum_{i=1}^{n} y_i^2(t), \qquad \sum_{i=1}^{n} y_i^2(\sigma_j - \tau) \leqslant \sum_{i=1}^{n} y_i^2(t) \qquad (j = 1, \ldots, n) \tag{2.8}$$

We shall determine the least upper bound of the factor r occurring in the right-hand side of equation (2.6). It is obvious that this bound is attained on the boundary of the region, and that for its determination one may replace the inequalities (2.8) by the corresponding equalities.

Let us make the following change of variables in the expression (2.7):

$$y_i = r\gamma_i \qquad (i = 1, \ldots, n) \qquad (r = \sqrt{y_1^2 + \ldots + y_n^2}) \tag{2.9}$$

where $\gamma_1, \ldots, \gamma_n$ are the direction cosines of the radius vector of a point on the surface $y_1^2 + \ldots + y_n^2 = r^2$. We thus obtain the equation

$$U = r^2 \sum_{i,\, j,\, \mu,\, \nu,\, s=1} c_{\mu\nu}(t) k_{\mu i}(t) \gamma_i(t) [p_{\nu s}(\sigma_\nu) k_{sj}(\sigma_\nu) \gamma_j(\sigma_\nu) + q_{\nu s}(\sigma_\nu) k_{sj}(\sigma_\nu - \tau) \gamma_j(\sigma_\nu - \tau)]$$

Since $|\gamma_i| \leqslant 1$, it follows that in the region (2.8) the next inequality holds:

$$\sup U < r^2 \sum_{i,\, j,\, \mu,\, \nu,\, s=1}^{n} |c_{\mu\nu}(t)|\, |k_{\mu i}(t)|\, [|p_{\nu s}(\sigma_\nu)|\, |k_{sj}(\sigma_\nu)| + |q_{\nu s}(\sigma_\nu)|\, |k_{sj}(\sigma_\nu - \tau)|] \tag{2.10}$$

Obviously, inf U = $-$sup U under condition (2.7). Therefore, the derivative dV/dt must satisfy the inequality

$$\frac{dV}{dt} < \sum_{i,\,j=1}^{n} a_{ij}^{(1)}(t)\, y_i(t)\, y_j(t) +$$

$$+ \tau \sum_{i=1}^{n} y_i^2(t) \Big\{ \sum_{i,\,j,\,\mu,\,\nu,\,s=1} |c_{\mu\nu}(t)|\; |k_{\mu i}(t)|\, |p_{\nu s}(\sigma_\nu)|\, |k_{sj}(\sigma_\nu)| +$$

$$+ |q_{\nu s}(\sigma_\nu)|\, |k_{sj}(\sigma_\nu - \tau)|\,|\Big\}$$

Setting

$$\omega(t, \tau) = \sup \sum_{i,\,j,\,\mu,\,\nu,\,s=1} |c_{\mu\nu}(t)|\; |k_{\mu i}(t)|\, [|p_{\nu s}(\sigma_\nu)| \cdot |k_{sj}(\sigma_\nu)| +$$

$$+ |q_{\nu s}(\sigma_\nu)|\; |k_{sj}(\sigma_\nu - \tau)|\,] \tag{2.11}$$

in the region $t - \tau \leqslant \sigma_\nu \leqslant t$ ($\nu = 1, \ldots, n$) we obtain

$$\frac{dV}{dt} < \sum_{i,\,j=1}^{n} a_{ij}^{(1)}(t)\, y_i y_j + \tau\omega(t, \tau) \sum_{i=1}^{n} y_i^2$$

or

$$\frac{dV}{dt} < \sum_{i,\,j=1}^{n} [a_{ij}^{1}(t) + \delta_{ij}\tau\omega(t, \tau)]\, y_i y_j, \quad \delta_{ij} \text{ is Kronecker's delta} \tag{2.12}$$

Theorem 2. The trivial solution of the system of differential equations of the first approximation is asymptotically stable if the roots $\lambda_1(t, \tau), \ldots, \lambda_n(t, \tau)$ of the equation

$$\det \| a_{ij}^{(1)}(t) + \delta_{ij}(\tau\omega(t, \tau) - \lambda) \| = 0$$

satisfy the condition $\lambda_i(t, \tau) \leqslant \epsilon < 0$ $(i = 1, \ldots, n)$ $t > t_0$, where ϵ is a negative number of arbitrarily small absolute value.

It is thus clear that there always exists a non-zero value of the lag τ for which the asymptotic stability of the solution of the system (1.4) implies the asymptotic stability of the system (1.3).

3. Theorems 5, 6 and 7 of [2] permit us to obtain not only sufficient conditions for the asymptotic stability, as formulated in our Theorems 1 and 2, but they also yield estimates of the disturbances. Let us assume that the obtained estimate is of the form:

$$V \leqslant V_0 \exp \int_{t_0}^{t} \lambda'(t)\, dt \tag{3.1}$$

The inequality (3.1) must then be implied by the values of the derivative dV/dt which depend on the coordinates $x_1(t), \ldots, x_n(t)$, $x_1(t-\tau), \ldots, x_n(t-\tau)$ of two points lying on an integral curve. Because of (3.1) these points must satisfy the inequality

$$V(t, x_1(t), \ldots, x_n(t) \exp \int_{t-\tau}^{t} -\lambda'(t)\,dt \leqslant V(t-\tau, x_1(t-\tau), \ldots, x_n(t-\tau)) \quad (3.2)$$

Next, let the values of $x_1(t), \ldots, x_n(t)$ be kept fixed. Under this condition we shall determine the sup dV/dt in the region of values of $x_1(t-\tau), \ldots, x_n(t-\tau)$ given by the inequality

$$V(t-\tau, x_1(t-\tau), \ldots, x_n(t-\tau)) \leqslant V(t, x_1(t), \ldots, x_n(t)) \exp \int_{t-\tau}^{t} -\lambda'(t)\,dt \quad (3.3)$$

Constructing the function

$$\Phi_1 = \frac{dV}{dt} - \nu\Big[\sum_{i=1}^{n} y_i^2(t-\tau) - \varphi(t)\sum_{i=1}^{n} y_i^2(t)\Big] \qquad \Big(\varphi(t) = \exp \int_{t-\tau}^{t} -\lambda'(t)\,dt\Big) \quad (3.4)$$

and making computations analogous to those that were performed in Section 1, we obtain the following inequality for the derivative dV/dt:

$$\sum_{i,j=1}^{n} (\beta_{ij}(t) + \delta_{ij}\sqrt{\varphi(t)h'(t)})\,\xi_i\xi_j \geqslant \frac{dV}{dt}\frac{d}{dt}\sum_{i=1}^{n}\xi_i^2 \geqslant$$
$$\geqslant \sum_{i,j=1}^{n} (\beta_{ij}(t) + \delta_{ij}\sqrt{\varphi(t)h''(t)})\,\xi_i\xi_j \quad (3.5)$$

where $\xi_1^2 + \ldots + \xi_n^2 = V$ and the variables $\xi_1, \ldots, \xi_n$ are connected with the variables $x_1, \ldots, x_n$ by the same transformation formulas as those in Section 1.

The function $\phi(t)$ has to be such that the equation (3.5) must imply the assumed estimate (3.1). Hence, for every $t \geqslant t_0$, the function $\lambda'(t)$ must not exceed the largest root of the equation

$$\det\|\beta_{ij}(t) + \delta_{ij}(\sqrt{\varphi(t)h'} - \lambda)\| = 0$$

Suppose $\mu'(t)$ is the largest root of the equation

$$\det\|\beta_{ij}(t) - \delta_{ij}\mu\| = 0.$$

Then it is obvious that $\lambda'(t)$ will have to be determined from the inequality

$$\lambda'(t) \geqslant \mu'(t) + \sqrt{\varphi(t)h'(t)} \quad (3.6)$$

But in accordance with the definition of $\phi(t)$, the function $\lambda'(t)$ is actually determined by the inequality (3.6).

The inequality (3.6) can be replaced by an equality. Replacing $\phi(t)$

by its value from (3.4) we obtain

$$\lambda'(t) = \mu'(t) + \sqrt{h'(t)}\exp\left(-\tfrac{1}{2}\int_{t-\tau}^{t} \lambda'(t)\,dt\right) \tag{3.7}$$

The disturbance can thus be found in the form (3.1) where the function $\lambda'(t)$ is determined by (3.6) or (3.7).

In an analogous manner one can obtain an estimate for the disturbance in the case when the estimate of dV/dt is found on the basis of Theorems 6 and 7 of [2].

Let us suppose again that the resulting estimate has the form

$$V \leqslant V_0 \exp \int_{t_0}^{t} \lambda_1(t)\,dt \tag{3.8}$$

Performing the orthogonal transformation of variables to the new variables $y_1, \ldots, y_n$, we obtain, for $\sigma < t$,

$$\sum_{i=1}^{n} y_i^2(\sigma) \leqslant \sum_{i=1}^{n} y_i^2(t)\Big) \exp \int_{\sigma}^{t} -\lambda_1(t)\,dt \tag{3.9}$$

The inequality (3.9) implies that the variables

$$y_1(\sigma_\nu), \ldots, y_n(\sigma_\nu), \qquad y_1(\sigma_\nu - \tau), \ldots, y_n(\sigma_n - \tau)\ (\nu = 1, \ldots, n)$$

which enter in the expression (2.6) for dV/dt, satisfy the inequalities

$$\sum_{j=1}^{n} y_i^2(\sigma_j) \leqslant \left(\exp \int_{\sigma_j}^{t} -\lambda_1(t)\,dt\right) \sum_{i=1}^{n} y_i^2(t)$$

$$\sum_{i=1}^{n} y_i^2(\sigma_j - \tau) \leqslant \left(\exp \int_{\sigma_j-\tau}^{t} -\lambda_1(t)\,dt\right) \sum_{i=1}^{n} y_i^2(t) \tag{3.10}$$

where

$$\sigma_j = t - \theta_j\tau \qquad 0 < \theta_j < 1$$

Setting

$$\varphi_j(t) = \exp \int_{t-\theta_j\tau}^{t} -\lambda_1(t)\,dt, \quad \psi_j(t) = \exp \int_{t-(1+\theta_j)\tau} -\lambda_1(t)\,dt$$

$$\varphi(t) = \exp \int_{t-\tau}^{t} -\lambda_1(t)\,dt, \ \psi(t) = \exp \int_{t-2\tau}^{t} -\lambda_1(t)\,dt$$

and assuming that $\lambda_1(t) \leqslant 0$, we obtain

$$\varphi_j(t) \leqslant \varphi(t), \qquad \psi_j(t) \leqslant \psi(t) \qquad (j = 1, \ldots, n) \tag{3.11}$$

In this connection it should be noted that the inequalities (3.10) can be strengthened and given the form

$$\sum_{i=1}^{n} y_i^2(\sigma_j) \leqslant \varphi(t) \sum_{i=1}^{n} y_i^2(t), \qquad \sum_{i=1}^{n} y_i^2(\sigma_j - \tau) \leqslant \psi(t) \sum_{i=1}^{n} y_i^2(t) \qquad (3.12)$$

One can then determine the sup dV/dt in the region (3.12).

Performing computations analogous to those made in Section 2, we obtain for dV/dt the inequality (sup $U = -$ inf U)

$$\sum_{i,j=1}^{n} \{a_{ij}^{(1)}(t) - \delta_{ij}[\omega_1(t,\tau)\sqrt{\varphi(t)} + \omega_2(t,\tau)\sqrt{\psi(t)}]\tau\} y_i y_j < \frac{dV}{dt} <$$

$$< \sum_{i,j=1}^{n} \{a_{ij}^{(1)}(t) + \delta_{ij}[\omega_1(t,\tau)\sqrt{\varphi(t)} + \omega_2(t,\tau)\sqrt{\psi(t)}]\tau\} y_i y_j \qquad (3.13)$$

where

$$\omega_1(t,\tau) = \sup \sum_{i,j,\mu,\nu,s=1}^{n} |c_{\mu\nu}(t)\, k_{\mu i}(t)\, p_{\nu s}(\sigma_\nu)\, k_{sj}(\sigma_\nu)|$$

$$\omega_2(t,\tau) = \sup \sum_{i,j,\mu,\nu,s=1} |c_{\mu\nu}(t)\, k_{\mu i}(t)\, q_{\nu s}(\sigma_\nu)\, k_{sj}(\sigma_\nu - \tau)|$$

in the region $t - r \leqslant \sigma_\nu \leqslant t$.

The functions $\phi(t)$ and $\psi(t)$ must be such that for every $t > t_0$ the function $\lambda_1(t)$ be not smaller than the largest root of the equation

$$\det \| a_{ij}^{(1)}(t) + \delta_{ij}[\tau(\omega_1(t,\tau)\sqrt{\varphi(t)} + \omega_2(t,\tau)\sqrt{\psi(t)}) - \lambda] \| = 0$$

Let $\mu_1(t)$ be the largest root of the equation

$$\det \| a_{ij}^{(1)} - \delta_{ij}\mu \| = 0$$

Then, obviously,

$$\lambda_1(t) \geqslant \mu_1(t) + \tau\Big[\omega_1(t,\tau)\exp\tfrac{1}{2}\int_{t-\tau}^{t} -\lambda_1(t)\,dt +$$

$$+ \omega_2(t,\tau)\exp\tfrac{1}{2}\int_{t-2\tau}^{t} -\lambda_1(t)\,dt\Big] \qquad (3.14)$$

The inequality (3.14) determines the function $\lambda_1(t)$. Thus the estimate of the form (3.8) can be obtained, and the function $\lambda_1(t)$ can be determined by means of the inequality (3.14).

For the determination of the function $\lambda'(t)$ by means of (3.7) and of $\lambda_1(t)$ by the use of (3.14), one can apply the well known method of successive approximations. (It should be noted that (3.14) may be replaced by an equality).

Let $\lambda_0'(t)$, $\lambda_1'(t)$, ..., $\lambda_n'(t)$, ..., and $\lambda_1^{(0)}(t)$, $\lambda_1^{(1)}(t)$, ..., $\lambda_1^{(n)}(t)$, ... be approximating sequences for $\lambda'(t)$ and $\lambda_1(t)$ respectively.

Then, assuming that $\lambda_0' \equiv 0$, or $\lambda_1^{(0)} = 0$, we obtain

$$\lambda_1'(t) = \mu'(t) + \sqrt{h'(t)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_n'(t) = \mu'(t) + \sqrt{h'(t)}\exp\left(\frac{1}{2}\int_{t-\tau}^{t} -\lambda_{n-1}'(t)\,dt\right)$$

$$\lambda_1^{(1)}(t) = \mu_1(t) + \tau\,[\omega_1(t,\tau) + \omega_2(t,\tau)]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_1^{(n)}(t) = \mu_1(t) + \tau\left[\omega_1(t,\tau)\exp\frac{1}{2}\int_{t-\tau}^{t} -\lambda_1^{(n-1)}(t)\,dt + \right.$$
$$\left. + \omega_2(t,\tau)\exp\frac{1}{2}\int_{t-2\tau}^{t} -\lambda_1^{(n-1)}(t)\,dt\right]$$

Questions on the convergence of the thus obtained sequences are not considered in this work.

4. Returning to the original systems (1.1) and (1.2) we can formulate the following sufficiency criteria for stability.

Theorem 3. Let the system of differential equations of first approximations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n}(p_{ij}(t)\,x_j(t) + q_{ij}(t)\,x_j(t-\tau)) \qquad (i = 1,\ldots,n)$$

be given. Let there exist, for this system, a positive definite Liapunov function whose derivative is dominated by a negative definite quadratic form. Then the perturbed motion is asymptotically stable and independent of the functions X_i.

The proof of this theorem is analogous to the proof of the corresponding theorem for systems without lag [7].

It is obvious that Theorem 3 is true also if the functions X_i are restricted, for example, by conditions [7] of the type

$$|X_i(t; x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau))| <$$
$$< A\{|x_1(t)| + \ldots + |x_n(t)| + |x_1(t-\tau)| + \ldots + |x_n(t-\tau)|\}$$

where A is a sufficiently small constant.

On the basis of Theorem 3 one can assert that the conditions, given in Theorems 1 and 2 for the asymptotic stability of the null solution of the first approximation equations, are also sufficient conditions for the asymptotic stability of the original system (1.1).

5. As an illustrative example let us consider the second order differential equation

$$\ddot{\varphi}(t) + a_1\dot{\varphi}(t) + a_2\varphi(t) + a_3\varphi(t-\tau) = 0 \tag{5.1}$$

which describes transient processes in certain automatic control systems [8].

We introduce the notation

$$\dot{\varphi}(t) = x_1(t), \qquad \varphi(t) = x_2(t), \qquad a_i = -b_i \qquad (i = 1, 2, 3)$$

Then the equation (5.1) can be written in the form of the system

$$\frac{dx_1}{dt} = b_1x_1(t) + b_2x_2(t) + b_3x_2(t-\tau), \ \frac{dx_2}{dt} = x_1(t) \tag{5.2}$$

If we let $\tau = 0$, we obtain the system

$$\frac{dx_1}{dt} = b_1x_1(t) + (b_2+b_3)x_2(t), \qquad \frac{dx_2}{dt} = x_1(t) \tag{5.3}$$

Let us suppose that the trivial solution of the system (5.3) is stable. Under this assumption we shall try to determine the Liapunov function as a quadratic form

$$V = \alpha_{11}x_1^2 + 2\alpha_{12}x_1x_2 + \alpha_{22}x_2^2$$

satisfying the equation

$$dV/dt = -2(x_1^2 + x_2^2)$$

where the derivative dV/dt is computed on the basis of (5.3). Solving this equation we obtain

$$\alpha_{11} = \frac{1-(b_2+b_3)}{b_1(b_2+b_3)}, \qquad \alpha_{12} = -\frac{1}{b_2+b_3}, \qquad \alpha_{22} = \frac{b_1^2 + (b_2+b_3)^2 - (b_2+b_3)}{b_1(b_2+b_3)} \tag{5.4}$$

Under the hypothesis that

$$a_1 > 0, a_2 + a_3 > 0$$

the quadratic form V will be positive definite. Let us evaluate its derivative on the basis of (5.2). We thus find that

$$\frac{1}{2}\frac{dV}{dt} = [(\alpha_{11}b_1 + \alpha_{12})x_1^2(t) + (\alpha_{11}b_2 + \alpha_{12}b_1 + \alpha_{22})x_1(t)x_2(t) + \alpha_{12}b_2x_2^2(t)] + \\ + b_3[\alpha_{11}x_1(t) + \alpha_{12}x_2(t)]x_2(t-\tau) \tag{5.5}$$

According to Section 1, one can obtain the stability conditions as the conditions that the function dV/dt be negative if $V(x_1(t-\tau), x_2(t-\tau)) \leqslant V(x_1(t), x_2(t))$. For the purpose of finding a dominating function for dV/dt we transform V to the canonical form by the substitutions

$$x_1 = k_1 y_1 + k_2 y_2, \qquad x_2 = y_2 \qquad \left(k_1 = \frac{1}{\alpha_{11}} \sqrt{\alpha_{11}\alpha_{22} - \alpha_{12}^2},\ k_2 = \frac{\alpha_{12}}{\alpha_{11}}\right)$$

Then

$$V = \frac{1}{\alpha_{11}} (\alpha_{11}\alpha_{22} - \alpha_{12}^2)(y_1^2 + y_2^2)$$

Expressing the equation (5.5) in terms of the variables y_1, y_2, we obtain

$$\frac{1}{2}\frac{dV}{dt} = (\alpha_{11}b_1 + \alpha_{12}) k_1^2 y_1^2(t) + [(\alpha_{11}b_1 + \alpha_{12}) 2k_1k_2 + $$
$$+ (\alpha_{11}b_2 + \alpha_{12}b_1 + \alpha_{22}) k_1]\, y_1(t)\, y_2(t) + [(\alpha_{11}b_1 + \alpha_{12}) k_2^2 + $$
$$+ (\alpha_{11}b_2 + \alpha_{12}b_1 + \alpha_{22}) k_2 + \alpha_{12}b_2]\, y_2^2(t) + b_3\alpha_{11}k_1 y_1(t)\, y_2(t-\tau)$$

The largest value that dV/dt can attain in the region

$$y_1^2(t-\tau) + y_2^2(t-\tau) \leqslant y_1^2(t) + y_2^2(t)$$

cannot exceed sup dV/dt in the region $|y_2(t-r)| \leqslant |y_1(t)| + |y_2(t)|$.

The function dV/dt will be negative definite if those quadratic forms are negative definite, which are obtained from dV/dt by setting $y_2(t-r)$ equal to $y_1 \pm y_2$, or $-y_1 \pm y_2$. Noting that the first Sylvester inequality is always satisfied, we obtain the following criterion for the definiteness of the two mentioned quadratic forms:

$$[(\alpha_{11}b_1 + \alpha_{12}) + (\alpha_{11}/k_1)|b_3|][\alpha_{11}b_1 + \alpha_{12})\ k_2^2 + (\alpha_{11}b_2 + \alpha_{12}b_1 + \alpha_{22})k_2 + \alpha_{12}b_2] -$$
$$- \frac{1}{4}[(\alpha_{11}b_1 + \alpha_{12}) 2k_2 + (\alpha_{11}b_2 + \alpha_{12}b_1 + \alpha_{22}) \pm \alpha_{11}b_3]^2 > 0$$

Expressing a_{11}, a_{12}, a_{22}, k_1, k_2, b_1, b_2, b_3 in terms of the coefficients a_1, a_2, a_3 of the original system, we find that

$$[1 - |a_3|(1 + a_2 + a_3)^2 / a_1(a_2 + a_3)\sqrt{(a_2 + a_3)[(1 + a_2 + a_3)^2 + a_1^2]}] \times [1 + a_1^2/(1 + a_2 +$$
$$+ a_3)^2] - [a_1/(1 + a_2 + a_3) + |a_3|(1 + a_2 + a_3)/a_1(a_2 + a_3)]^2 > 0 \qquad (5.7)$$

Thus we have obtained conditions for the stability of the trivial solution of the equation (5.1) which are independent of the value of the lag r ; i.e. conditions (5.7), and $a_1 > 0$, $a_2 + a_3 > 0$.

Making use of Theorem 2, one can obtain stability criteria that involve the lag r.

Let us assume that in accordance with equation (2.2) the following condition is satisfied for (5.5)

$$x_2(t-\tau) = x_2(t) - \tau \frac{d}{dt} x_2(\sigma) \qquad (\sigma = t - \theta\tau,\ 0 < \theta < 1)$$

From (5.2) we have: $x_2(t-r) = x_2(t) - r x_1(\sigma)$. Making the proper substitutions in (5.5) we obtain

$$\frac{1}{2}\frac{dV}{dt} = -(x_1^2(t) + x_2^2(t)) - b_3\tau [\alpha_{11}x_1(t) + \alpha_{12}x_2(t)]\, x_1(\sigma) \qquad (5.8)$$

The conditions for stability can also be expressed as conditions that the function dV/dt be negative if

$$V(x_1(\sigma), x_2(\sigma)) \leqslant V(x_1(t), x_2(t))$$

Let us transform the quadratic form V to its canonical form by means of the substitutions

$$x_1 = y_1, \quad x_2 = l_1 y_1 + l_2 y_2, \quad l_1 = -\frac{a_{12}}{a_{22}}, \quad l_2 = \frac{1}{\alpha_{22}}\sqrt{\alpha_{11}\alpha_{22} - \alpha_{12}^2}$$

Then

$$V = \frac{1}{\alpha_{22}}(\alpha_{11}\alpha_{22} - \alpha^2_{12})(y_1^2 + y_2^2)$$

$$\frac{1}{2}\frac{dV}{dt} = -\{y_1^2(t) + l_1^2 y_1^2(t) + 2l_1 l_2 y_1(t) y_2(t) + l_2^2 y_2^2(t)] - \\ - b_3\tau[(\alpha_{11} + \alpha_{12}l_1) y_1(t) + \alpha_{12}l_2 y_2(t)] y_1(\sigma)$$

The least upper bound of the derivative dV/dt in the region

$$y_1^2(\sigma) + y_2^2(\sigma) \leqslant y_1^2(t) + y_2^2(t)$$

cannot exceed the least upper bound of dV/dt in the region

$$|y_1(\sigma)| \leqslant |y_1(t)| + |y_2(t)|$$

Analogously to the above procedure we obtain as a sufficient condition for dV/dt to be negative, the condition of positiveness of the quadratic form

$$[(1 + l_1^2) - |b_3|\tau(\alpha_{11} + \alpha_{12}l_1)] y_1^2 + [l_2^2 - |b_3|\tau\alpha_{12}l_2] y_2^2 + [2l_1 l_2 + \\ + |b_3 + \tau(\alpha_{11} + \alpha_{12}l_1 + \alpha_{12}l_2)] y_1 y_2 \qquad (5.9)$$

that is, the following condition

$$[1 + l_1^2 - |b_3|\tau(\alpha_{11} + \alpha_{12}l_1)][l^2_2 - |b_3|\tau\alpha_{12}l_2] - [l_1 l_2 + {}^1/_2 |b_3|\tau(\alpha_{11} + \alpha_{12}l_1 + \alpha_{12}l_2)]^2 > 0 \quad (5.10)$$

Thus, on the basis of Theorem 2 we find that the trivial solution of (5.1) will be stable if $a_1 > 0$, $a_2 + a_3 > 0$ and condition (5.10) is satisfied.

Beginning with a certain value τ, the region of stability determined by the inequality (5.10) either intersects or lies entirely within the stability region determined by the inequality (5.7).

This shows that the estimate of dV/dt by the method of Section 2 is more exact than the one given in Section 1 for small values of the lag only.

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